Research Paper | Education | E-ISSN No : 2454-9916 | Volume : 10 | Issue : 10 | October 2024

EXPLORING PROOFS OF THE PYTHAGOREAN THEOREM

Rishan Bheda

Research Scholars Program, Harvard Student Agencies, In collaboration with Learn with Leaders

ABSTRACT

The Pythagorean Theorem has been a cornerstone of mathematics since its discovery, offering profound insights and practical applications across various fields. This paper explores the historical origins and numerous proofs of the theorem, emphasizing its enduring significance. Through an analysis of classical and modern proof techniques, including dissection proofs, similar triangle proofs, and newly discovered frame arrangements, this study highlights the theorem's versatility and impact. Additionally, the paper discusses the wide-ranging applications of the Pythagorean Theorem in contemporary society, from computer graphics and navigation to physics and engineering, demonstrating its essential role in both theoretical and practical contexts.

KEYWORDS: Pythagorean Theorem, Proofs, Geometry, Frames, Navigation, Physics

INTRODUCTION

Since its discovery, the Pythagorean Theorem has remained a cornerstone of mathematics, propelling advancements and deepening our understanding of the field. As stated by the German astronomer and mathematician Johannes Kepler (1596) in his cosmological work *Mysterium Cosmographicum*, "Geometry has two great treasures: One is the Theorem of Pythagoras and the other division of line into mean and extreme ratio. The first we may compare to a measure of gold; the second we may name a precious jewel."

Although archaeological evidence indicates that the Egyptians, Babylonians, and Chinese discovered the theorem before Pythagoras, it is generally credited to Pythagoras, a Greek mathematician, for being the first to bring widespread awareness to it. The Pythagorean Theorem has numerous practical applications that have significantly contributed to the development of tools and machinery, shaping our modern world. It also serves as a basis for many other mathematical laws. The theorem has been the subject of extensive study, and approximately 400 different proofs are known today. This paper explores some of these proofs, particularly focusing on two recently discovered proofs.

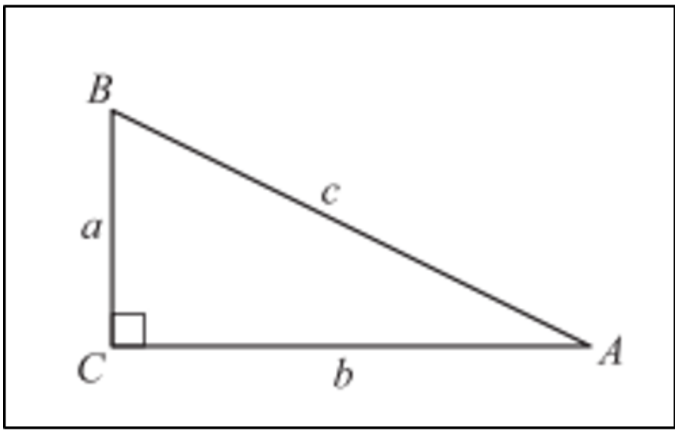

Figure 1: Right Triangle
Ratios Involved with a Trigonometric Function
(Unacademy, 2024)

In Euclidean geometry, the Pythagorean Theorem states that if the two legs of a right triangle have lengths a and b, and c is the length of the hypotenuse, the side opposite the right angle, then the sum of the areas of the squares on the legs is equivalent to the area of the square on the hypotenuse.

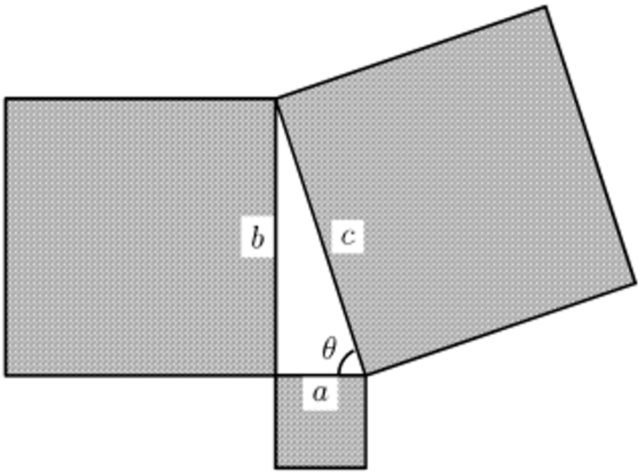

Figure 2: Pythagorean Triples
Agarwal (2020)

This leads to the infamous equation:

$$a^2+b^2=c^2$$

This paper will explore a proof technique of the Pythagorean Theorem that uses four congruent framing triangles, as shown in Figure 1, to derive the theorem using a diagram. The small frame model and the large frame model are the two standard framing models, illustrated in Figures 3.1 and 3.2.

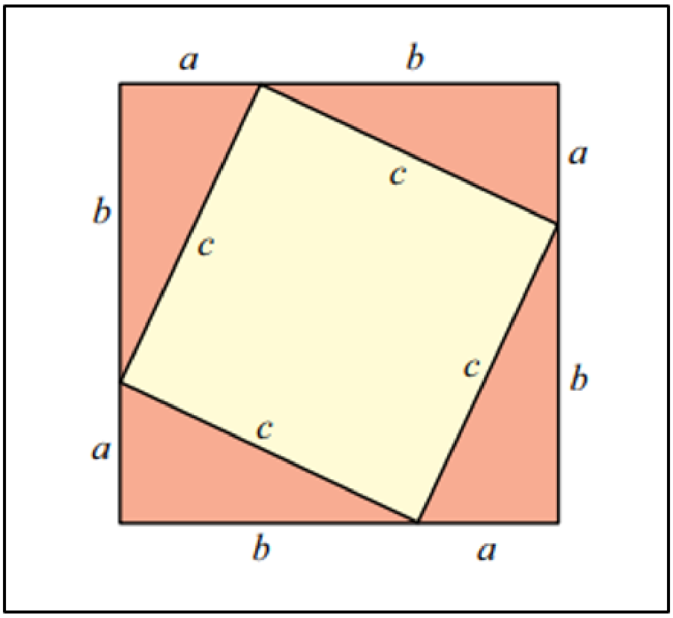

Figure 3.1: Small Frame Model

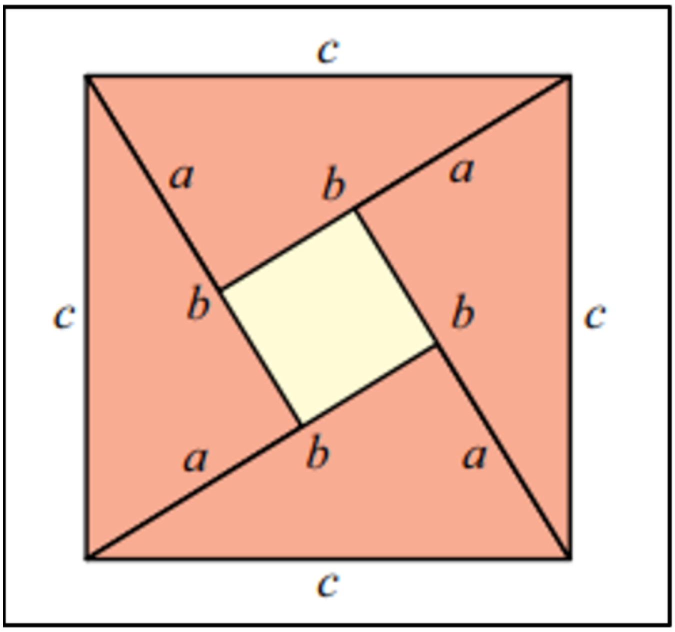

Figure 3.2: Large Frame Model
Source: Alexander Bogomolny (2018)

Brief History of the Theorem

The principles of the Pythagorean Theorem, although named after Pythagoras, can be traced back to civilizations before the ancient Greeks. Many historians suggest that the ancient Babylonians possessed knowledge of this fundamental right triangular relationship. A major artifact supporting this notion is the Plimpton 322 tablet (*The Babylonian tablet Plimpton 322, Bill Casselman*), dating back to around 1800 BCE. This tablet features meticulous records of various Pythagorean triples, presenting integer solutions to the equation a2+b2=c2a^2 + b^2 = c^2a2+b2=c2.

Similarly, the Yale YBC 7289 tablet (*BPOA 06, 0728 artifact entry (No. P210503). (2023, February 1). Cuneiform Digital Library Initiative (CDLI). https://cdli.ucla.edu/P210503*), also from around the same period, showcases a square with a side length of thirty, accompanied by an incredibly precise approximation of 2\sqrt{2}2, conveyed through Babylonian sexagesimal numerals. These artifacts indicate that the Babylonians had a sophisticated understanding of the relationships within right-angled triangles long before Pythagoras.

The Pythagoreans, followers of Pythagoras' teachings, are believed to have further developed this theorem in the 6th century BCE. Their contributions helped formalize and disseminate the theorem, leading to its enduring significance in mathematics.

History of Proofs through Frames

The framing proofs highlighted in the previous sections unveil captivating historical ties. Notably, the amalgamation of the "large and small frame" diagram, as depicted in Figure 5, traces its origins back to the ancient Chinese text known as Zhoubi Suanjing, believed to have been written between 100

BCE and 100 CE. Although primarily an astronomy text, the Zhoubi Suanjing acknowledges this principle as the Gougu rule (*Chinese Pythagorean theorem, page 22 Joseph Needham's Science and Civilization in China: Volume 3, 1986, Cave Books Ltd*). Proof number 253 in The Pythagorean Proposition demonstrates how the Pythagorean relationship can be affirmed through a similar diagram. Furthermore, the 12th-century Indian mathematician Bhaskara incorporated a diagram akin to the ancient Chinese version in his renowned work Lilavati (1150), further reinforcing the cross-cultural transmission and adaptation of mathematical concepts across civilizations.

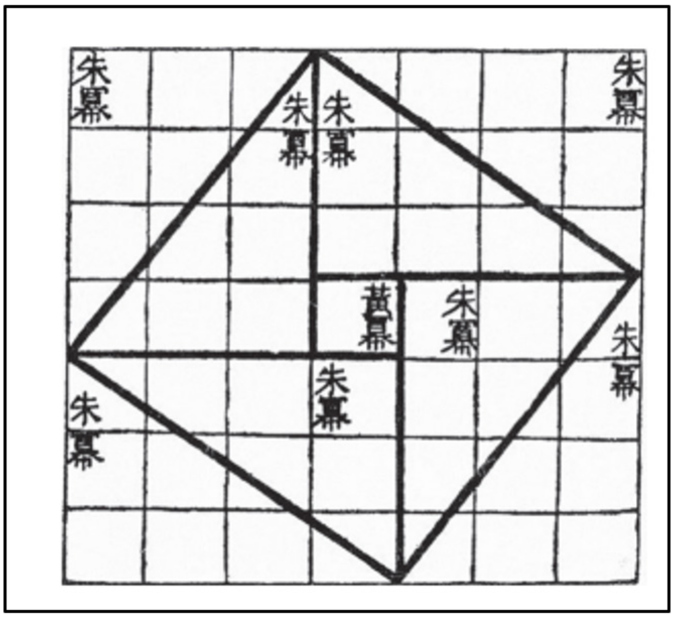

Figure 4: Large-small Frame Diagram
Source: Alexander Bogomolny (2018)

Another notable framing proof employs half of the large frame diagram arranged in a trapezoidal configuration, as shown in Figure 5. In 1876, prior to assuming office as the twentieth President of the United States, James Garfield, then an Ohio Representative, utilized a diagram akin to Figure 6 to establish the Pythagorean theorem. This finding was subsequently published in the New England Journal of Education ("Pons Asinorum". New England Journal of Education. 1876, ISSN 2578-4145).

Throughout history, numerous scholars have derived the Pythagorean theorem through various methods. It is important to mention that pinpointing the origins of these proofs is a rather imprecise science. While we may possess a general understanding of how different concepts evolved and who contributed to them, in certain instances, it can be arduous or nearly impossible to definitively ascertain the exact origins of a particular proof.

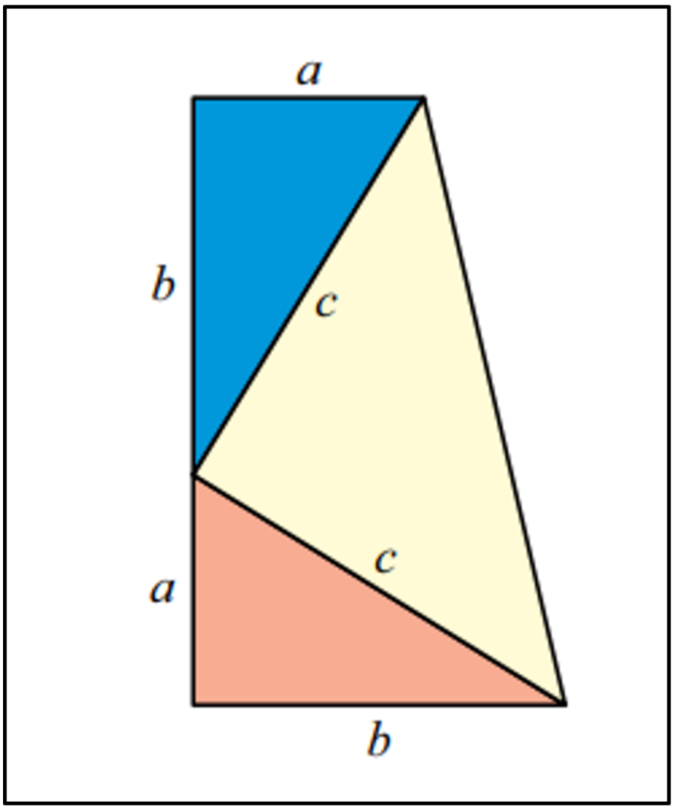

Figure 5: Trapezoidal Diagram

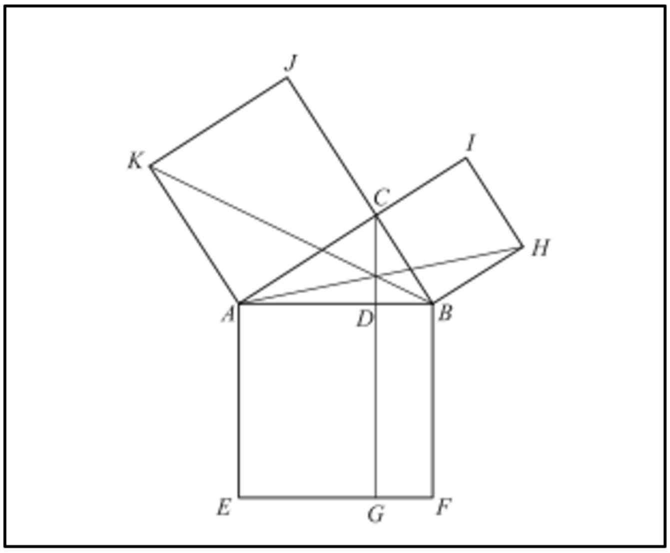

Figure 6: Euclid's Windmill

Some Proof Techniques

The Pythagorean Theorem offers multiple derivations, each categorized by its distinct proof techniques. This section provides a concise overview of the dissection and similar triangle proof methods before delving into the workings of the large and small frame proofs.

Dissection Proofs

Dissection proofs involve sophisticated arguments that deconstruct the square on the hypotenuse into regions that can be rearranged into the two squares on the legs. Euclid's renowned windmill diagram, depicted in Figure 6, serves as

an exemplary demonstration of such proof. Another elegant dissection argument, attributed to the 19th-century amateur British mathematician Henry Perigal, utilizes the diagram showcased in Figure 7. This diagram is dissected using vertical and horizontal lines from the midpoint of each side of the hypotenuse square, along with lines passing through the center of the longer leg's square, which are parallel and perpendicular to the hypotenuse. Further details regarding this derivation can be found in Casselman's article on Henry Perigal and his proof (*Bill Casselman, On the dissecting table (Sep 2001)*).

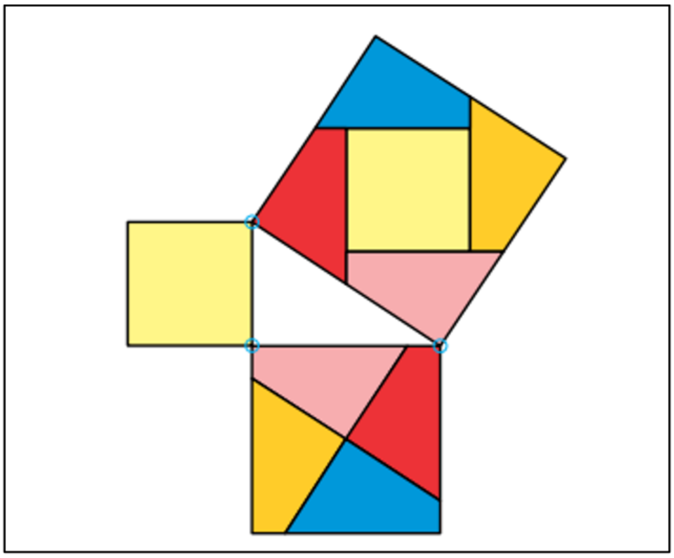

Figure 7: Perigal's Dissection Diagram
Source: Alexander Bogomolny (2018)

Small and Large Frame Proof

For the large frame proof, we use a standard right triangle, like the one shown in Figure 1, to build the pair of diagrams shown in Figure 8 using two sets of four congruent copies of the triangle.

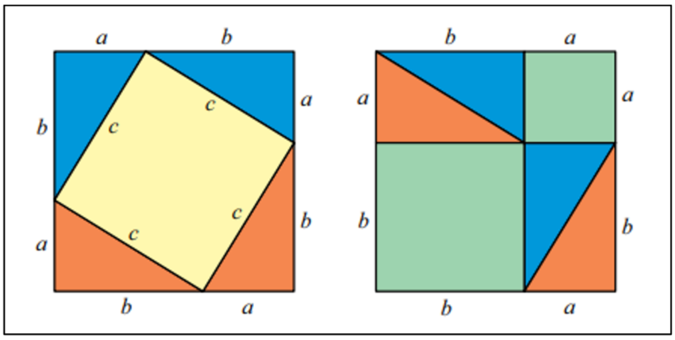

Figure 8: Large Frame Proof
Source: Alexander Bogomolny (2018)

Through the equal area argument, these two diagrams provide a visual demonstration of the fact that $a^2+b^2=c^2$. Using a more algebraic approach, we can equate the total area of the frame on the left with the sum of its five parts.

The area of the frame comes to be $(a+b)^2$, while the area of each triangle is $1/2\ ab$. This leads to

$$(a+b)^2=(4)\ \ 1/2\ ab+c^2\ \ a^2+2ab+b^2=2ab+c^2\ \ a^\wedge 2+b^2=c^2$$

These two diagrams provide another proof for the Pythagorean identity using the equal area argument. Once again, this can be presented using algebra. We have

$$c^2=(4)\ \ 1/2\ ab+(b-a)^2$$
$$=b^2+a^2$$

Similar Triangle Proof

Another type of proof of the theorem makes use of similar triangles. The diagram shown in Figure 9 is one form of this proof in which a larger right triangle is divided into two smaller right triangles by its altitude.

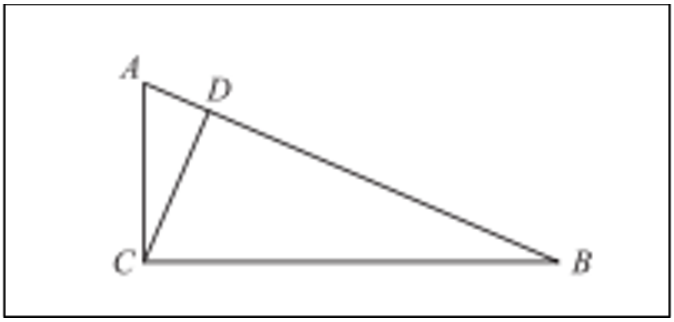

Figure 10
Source: Alexander Bogomolny (2018)

Two New Frames

The search for new ways to prove the Pythagorean Theorem is ongoing. After experimentation with different arrangements, two new methods were discovered by math professors Ian Adelstein and George Ashline: the 'NinjaSstar Frame' and the 'Overlapping Frame.' Both these arrangements seamlessly fit back into the original small frame arrangement.

Ninja Star Frame

To achieve the frame outlined in this proof, we overlap the initial four congruent triangles, forming a ninja star frame as shown in Figure 11. This arrangement cuts the triangles along their altitude, leading to triangles that are similar to the original ones. Since the green triangles form a small frame, we use an argument similar to the one used before to equate the area of the frame with the sum of the areas of the four congruent triangles and the square in the center of the frame.

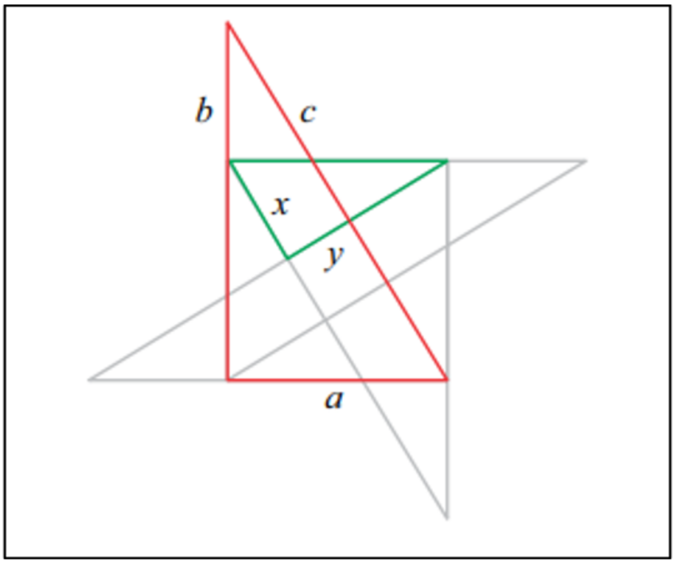

Figure 11: Ninja Star Frame
Source: Alexander Bogomolny (2018)

Overlapping Frame

Similarly, we examine four identical red replicas of the initial right triangle depicted in Figure 1. This time, we position them within a frame that intersects the four triangles, as illustrated in Figure 12. This layout divides each of the original red triangles along its altitude, generating groups of triangles that are similar to the initial ones. Among these sets of similar triangles, represented in blue in Figure 12, one adopts the small frame arrangement as seen in Figure 3. Consequently, the blue triangle fulfills the Pythagorean identity, and by virtue of similarity, the original red triangle does as well.

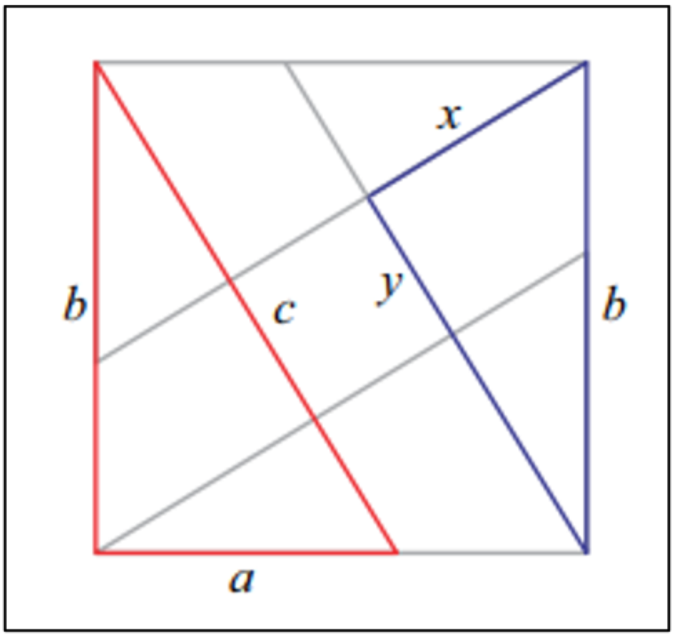

Figure 12: Overlapping frame
Source: Alexander Bogomolny (2018)

To prove this algebraically, we denote the lengths of the red triangle with a, b, and c and the corresponding lengths of the blue triangle with x, y, and b. As the blue and red triangles are similar, we get

$$\frac{a}{x} = \frac{b}{y} = \frac{c}{b} \text{ and } \frac{a^2}{x^2} = \frac{b^2}{y^2} = \frac{c^2}{b^2}$$

Using these inequalities, we can derive that

$$c^2 = \frac{c^2}{b^2} * b^2 = \frac{c^2}{b^2} * (x^2 + y^2) = \frac{c^2}{b^2} * x^2 + \frac{c^2}{b^2} * y^2 = \frac{a^2}{x^2} * x^2 + \frac{b^2}{y^2} * y^2 = a^2 + b^2$$

Consequences of Pythagoras' Theorem

The Pythagorean Theorem is a fundamental concept in mathematics, offering timeless illustrations of geometric understanding and deductive reasoning. Beyond its inherent mathematical value, the theorem's real-world applications have significantly impacted contemporary society. Numerous inventions and advancements in fields ranging from science and technology to architecture and engineering have their roots in the Pythagorean Theorem. In computer graphics, the theorem is essential for rendering images and creating virtual environments. It helps in calculating distances between points in a digital space, which is crucial for accurate image representation and animation. In navigation, the Pythagorean Theorem is used to determine the shortest path between two points, aiding in the development of GPS technology and route planning.

The field of physics also relies heavily on the Pythagorean Theorem. It is used to resolve vector quantities into their components, which is fundamental in understanding forces and motion. Similarly, in architecture and engineering, the theorem aids in designing and constructing buildings, bridges, and other structures. It ensures structural integrity by helping engineers calculate loads and stresses accurately.

Furthermore, the theorem's impact extends beyond academia into daily life. It plays a crucial role in design procedures, such as creating right-angled objects and determining optimal layouts in construction projects. The Pythagorean Theorem also underpins various problem-solving techniques used in fields like data science and machine learning, where it helps in clustering algorithms and distance calculations.

In summary, the Pythagorean Theorem's lasting influence highlights its critical role in shaping the modern world and underscores the enduring value of mathematical concepts in our culture. Its applications across diverse disciplines demonstrate the theorem's versatility and importance in both theoretical and practical contexts.

CONCLUSION

The Pythagorean Theorem is a fundamental mathematical principle that has significantly influenced various scientific and practical domains. From its ancient origins to its contemporary applications, the theorem has consistently demonstrated its value in solving complex problems and advancing human knowledge. The diverse proofs, including the newly discovered ninja star and overlapping frames, underscore the theorem's versatility

and the continuous quest for understanding in mathematics. The widespread applications of the Pythagorean Theorem in fields such as computer graphics, navigation, physics, and engineering further highlight its importance in modern society. Ultimately, the Pythagorean Theorem's enduring relevance exemplifies the timeless nature of mathematical discoveries and their profound impact on our world

REFERENCES

1. V. N. Beryozin, The good old Pythagorean theorem, Quantum, Jan./Feb. 1994, 25-28, 60-61.
2. Adelstein, I. M., & Ashline, G. L. (2019). Reframing the Pythagorean Theorem. The College Mathematics Journal, 50(1), 28–35. https://doi.org/10.1080/07468342.2019.1557983
3. Maor, E. (2007). The Pythagorean Theorem: A 4000-Year History. Princeton, NJ: Princeton University Press.
4. Bogolmony, A. (2012). Pythagorean theorem and its many proofs, interactive mathematics miscellany and puzzles. https://www.cut-the-knot.org/pythagoras/index.shtml.
5. Kepler, J. (1596). Mysterium Cosmographicum de Admirabile Proportine Orbium Celestium
6. Bill Casselman (Sep 2001), On the dissecting table
7. Chinese Pythagorean theorem, page 22 Joseph Needham's Science and Civilization in China: Volume 3, 1986, Cave Books Ltd)
8. Agarwal, R. P. (2020) Pythagorean Triples before and after Pythagoras, https://doi.org/10.3390/computation8030062
9. Bogomolny, A. (2018) Pythagorean Theorem. https://www.cut-the-knot.org/pythagoras/index.shtml
10. Unacademy (2024) Learn about the Various Ratios Involved with a Trigonometric Function. Unacademy. Available at: https://unacademy.com/content/jee/study-material/mathematics/learn-about-the-various-ratios-involved-with-a-trigonometric-function/